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The partition function of the two-dimensional black hole conformal field theory

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Abstract: We compute the partition function of the conformal field theory on the two-dimensional euclidean black hole background using path-integral techniques. We show that the resulting spectrum is consistent with the algebraic expectations for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset conformal field theory construction. In particular, we find confirmation for the bound on the spin of the discrete representations and we determine the density of the continuous representations. We point out the relevance of the partition function to all string theory backgrounds that include an $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset factor.

Keywords: Conformal Field Models in String Theory, Black Holes in String Theory.

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1. Introduction

Recently, substantial progress was made in understanding non-compact Wess-Zumino-Witten models. In particular, the spectrum and correlation functions of string theory on AdS_3, the covering space of $\mathrm{SL}(2,\mathbb{R})$, were analysed in detail in [1, 2, 3]. The spectrum for the non-compact Wess-Zumino-Witten model was determined in [1], using intuition for long strings obtained from [4, 5] and the technical tool of spectral flow [6], thereby solving the long-standing problem of determining the correct Hilbert space for the $\mathrm{SL}(2,\mathbb{R})$ WZW-model [7]–[18]. Next, in [2], the computation of the free energy for string theory on AdS_3[1] by path-integral methods gave additional support to the spectrum proposed in [1]. Finally, in [3] the completeness of the Hilbert space was checked by computing various correlators (see also e.g. [22]–[26] for earlier work). This series of papers has answered important questions in non-compact Wess-Zumino-Witten theories and opened the road to a more extensive study of these models.

It is natural then to re-address some old questions. In particular, we can re-analyze [27] the toroidal partition function for the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset conformal field theory. It is an important task to write down the partition function for this background for several reasons. The $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ background [28]–[34] was identified and analysed as a two-

[1] See also e.g. [19, 20, 21].

dimensional euclidean black hole in [35]. Subsequently, it was employed in the construction of conformal field theories describing exact string propagation on curved spacetimes (see e.g. [36]). Moreover, many interesting string theory backgrounds contain $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ factors, for example in the context of singularities of Calabi-Yau manifolds [37], holographic duals for NS5-brane backgrounds [38, 39, 40], et cetera. In order to write down the toroidal partition function for string theory on these backgrounds, the most crucial ingredient is the black hole background partition sum. A precise treatment of the spectrum and the Hilbert space for the coset model was hitherto lacking.

In this letter, we address the computation of the partition function from a path-integral point of view. Our computation is technically close to the analysis of the free energy of string theory on AdS_3 in [2]. In section 2 we discuss the setup for our computation, discussing a few general features of coset CFT and toroidal partition functions. In section 3 we perform the actual computation and discuss the crucial ingredient of Ray-Singer torsion. Next, we analyse the result and show that it agrees with the algebraic expectations in section 4. We conclude and discuss applications in section 5.

2. The $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset toroidal partition function

We introduce the model in this section and pay some attention to the holonomies that will play a crucial role in the computation of the partition function. The general treatment of gauged Wess-Zumino-Witten models is well-known [41]–[44]. For a general group manifold G the Wess-Zumino-Witten action is:

$$S[g] = \frac{k}{2\pi}\int_{\mathrm{WS}} d^2z\, \mathrm{Tr}(\partial g^{-1}\bar{\partial} g) + \frac{ik}{12\pi}\int_{\mathrm{B}} \mathrm{Tr}(g^{-1}dg)^3\,, \tag{2.1}$$

where $g(z,\bar{z})$ is a group element, the level of the WZW-model is $k\in\mathbb{R}$, and B is a three-dimensional manifold with the worldsheet as a boundary. Our worldsheet is a two-torus T^2.

For compact group manifolds, the level k is in general quantised but for $\mathrm{SL}(2,\mathbb{R})$ we have $H^3(\mathrm{SL}(2,\mathbb{R}),\mathbb{R}) = 0$ so that the action is independent of our choice of manifold B for any real k. The Wess-Zumino-Witten model has an affine symmetry $G(z)\times G(\bar{z})$. We will gauge an axial abelian subgroup of the symmetry group $g\to hgh$, yielding the action:

$$S_{\mathrm{gauged}}[g;A] = S[g] - \frac{k}{\pi}\int_{\mathrm{WS}} d^2z\, \mathrm{Tr}(\bar{A}\bar{\partial} g g^{-1} + Ag^{-1}\partial g + g^{-1}\bar{A}gA + A\bar{A}) \tag{2.2}$$

with the one-form gauge field $A_{(1)}$ defined as $A_{(1)} = A\,dz + \bar{A}\,d\bar{z}$. Our gauged theory is anomaly free (see e.g. [45]).

Next we concentrate on the Lorentzian $\mathrm{SL}(2,\mathbb{R})$ group manifold. A standard parametrisation of the group elements g is in terms of Euler angles [34]:

$$g = e^{\frac{i}{2}\theta_L\sigma_2} e^{\frac{r}{2}\sigma_1} e^{\frac{i}{2}\theta_R\sigma_2}\,, \tag{2.3}$$

where $0\leq r\leq\infty, 0\leq\theta_L<2\pi, -2\pi\leq\theta_R<2\pi$. We gauge the axial U(1) symmetry $g\to hgh$ where $h=\exp(\frac{i}{2}\lambda\sigma_2)$. The coordinates shift under gauge transformations as

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$\theta_{L,R} \to \theta_{L,R} + \lambda$ and the gauge field transforms as

$$A \longrightarrow A + d\lambda\,. \tag{2.4}$$

The gauged WZW action (2.2) becomes:

$$\begin{aligned} S[r,\theta_R,\theta_L;A] = \frac{k}{2\pi}\int d^2z\Big(&\frac{1}{2}(\partial r\bar\partial r - \partial\theta_L\bar\partial\theta_L - \partial\theta_R\bar\partial\theta_R - 2\cosh r\bar\partial\theta_L\partial\theta_R) + \\ &+ A(\bar\partial\theta_R + \cosh r\bar\partial\theta_L) + \bar A(\partial\theta_L + \cosh r\partial\theta_R) - \\ &- A\bar A(\cosh r + 1)\Big)\,. \end{aligned} \tag{2.5}$$

In [35] it was shown, by integrating out the gauge field classically, that the coset has a cigar geometry that can be interpreted as a Euclidean black hole. Gauging a non-compact abelian subgroup would have resulted in the Lorentzian two-dimensional black hole.

The gauged theory can be re-written in terms of the sum of an $\mathrm{SL}(2,\mathbb{R})$ model and a $\mathrm{U}(1)$ model. We thereto introduce the coordinates $\theta = \frac{1}{2}(\theta_L - \theta_R)$ and $\tilde\theta = \frac{1}{2}(\theta_L + \theta_R)$. In terms of these coordinates the action can be written in the manifestly gauge invariant form (since shifts $\tilde\theta \to \tilde\theta + \lambda$ are compensated by shifts in the gauge field $A \to A + \partial\lambda$ and $\bar A \to \bar A + \bar\partial\lambda$):

$$\begin{aligned} S[r,\theta,\tilde\theta;A,\bar A] = \frac{k}{2\pi}\int d^2z\Big[&\frac{1}{2}\partial r\bar\partial r + (\cosh r - 1)\Big(\partial\theta\bar\partial\theta + (A - \partial\tilde\theta)\bar\partial\theta - (\bar A - \bar\partial\tilde\theta)\partial\theta\Big) - \\ &- (\cosh r + 1)(A - \partial\tilde\theta)(\bar A - \bar\partial\tilde\theta)\Big]. \end{aligned} \tag{2.6}$$

To re-write the action in product form, we first Hodge-decompose the gauge field on the torus as:

$$\begin{aligned} A &= \partial\rho_L + \frac{i}{2\tau_2}(u_1\bar\tau - u_2) \\ \bar A &= \bar\partial\rho_R - \frac{i}{2\tau_2}(u_1\tau - u_2)\,, \end{aligned} \tag{2.7}$$

where $\rho_L^* = \rho_R$ is well-defined on the torus and the holonomies u_1, u_2, parametrize the Wilson lines on the toroidal worldsheet with modular parameter τ. Ignoring the holonomies for now, we can follow the treatment on the sphere (as in e.g. [46]). We introduce the new variables $\rho = \frac{1}{2}(\rho_L - \rho_R)$, $\tilde\rho = \frac{1}{2}(\rho_L + \rho_R)$ and $\kappa = \theta + \rho$, $\tilde\kappa = \tilde\theta - \tilde\rho$, in terms of which the action becomes:

$$\begin{aligned} S[r,\kappa,\tilde\kappa;\rho,\tilde\rho] = \frac{k}{2\pi}\int d^2z\Big(&\frac{1}{2}\partial r\bar\partial r + (\cosh r - 1)\partial\kappa\bar\partial\kappa - (\cosh r + 1)\partial\tilde\kappa\bar\partial\tilde\kappa + \\ &+ (\cosh r - 1)(\partial\kappa\bar\partial\tilde\kappa - \bar\partial\kappa\partial\tilde\kappa)\Big) + \frac{k}{\pi}\int d^2z\;\partial\rho\bar\partial\rho\,. \end{aligned} \tag{2.8}$$

Since under a gauge transformation $\rho_{L,R} \to \rho_{L,R} + \lambda$ the fields $\kappa, \tilde\kappa$ and ρ do not transform, the above action is gauge invariant.

We can read (2.8) as the action for the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ model. Of course, on a toroidal worldsheet we need to take care in following the holonomies in the gauge field through the coordinate redefinitions. Note however that already for the spherical topology, subtleties arise that have hitherto been succesfully ignored. Indeed, the quantity κ is a linear combination of a real and imaginary field, but will nevertheless be treated as a real field [34, 46]. We will briefly return to the subtle issues of analytic continuation in the following, although we will not resolve all of them unambigously. The above decomposition was put to good use in [34] to argue for the spectrum of the CFT and to calculate the exact black hole background, and in [46] to compute the effective action and rederive the exact metric in a path-integral approach. We note that the holonomies of the gauge field have been transformed, via the field redefinitions, into non-trivial windings (over the two 1-cycles of the torus) for the matter fields $\kappa, \tilde{\kappa}$ and ρ. They will be crucial in the following.[2]

Before delving into the main part of the computation of the partition function we gauge-fix the action by choosing $\tilde{\rho} = 0$. We then need to include the ghost action

$$S_{\text{ghosts}}[b,c] = \frac{1}{\pi} \int d^2 z (b\bar{\partial} c + \bar{b}\partial\bar{c}) \,. \tag{2.9}$$

3. Computing the partition function

This section contains the core of the computation of the toroidal partition function. We will discuss the various techniques needed for the computation in some detail. Our computation owes a lot to the analysis in [2, 27]. Of course, since [2] computes the free energy of AdS_3 string theory while we are interested in the partition function on the euclidean black hole background, we need to adapt their computational techniques creatively.[3]

Our previous treatment of the model was in accord with standard conventions on Euler angles, but to make the computation of the partition function feasible, it is very useful to parametrize the $\mathrm{SL}(2,\mathbb{R})$ part of the model in terms of the coordinates introduced in [27]. After continuing the path integral to Euclidean signature to make it well defined (effectively transforming the model into the $\mathrm{SL}(2,C)/\,\mathrm{SU}(2)$ coset model – for discussions see [27] and [3]), the coordinate transformation becomes:

$$\begin{aligned} v &= \sinh\frac{r}{2} e^{i\kappa} \\ \bar{v} &= \sinh\frac{r}{2} e^{-i\kappa} \\ \phi &= i\tilde{\kappa} - \log\cosh\frac{r}{2} \,.. \end{aligned} \tag{3.1}$$

Writing the total action in terms of these variables results in:

$$\begin{aligned} S[\phi, v, \bar{v}; \rho; b, c] = \frac{k}{\pi} \int d^2 z \Big(\partial\phi\bar{\partial}\phi + (\partial\bar{v} + \bar{v}\partial\phi)(\bar{\partial} v + v\bar{\partial}\phi) \Big) + \\ + \frac{k}{\pi} \int d^2 z \; \partial\rho\bar{\partial}\rho + \int d^2 z (b\bar{\partial} c + \bar{b}\partial\bar{c}) \,. \end{aligned} \tag{3.2}$$

[2] Our toroidal treatment of the holonomies will naturally turn out to be equivalent to the BRST analysis of the gauge invariant states in [34].

[3] To avoid confusion, note that the temperature introduced in [2] is the temperature of AdS_3. The euclidean black hole is an analytically continued version of the Lorentzian black hole with a different time direction.

Note that the fields $\phi, \upsilon, \bar{\upsilon}$ and ρ have non-trivial holonomies. In order to perform the path integral we will decompose them in a periodic part and a holonomy part:

$$\begin{aligned}
\phi &= \hat{\phi} + \frac{1}{4\tau_2}\Big((u_1\bar{\tau} - u_2)z + (u_1\tau - u_2)\bar{z}\Big) \\
\upsilon &= \hat{\upsilon}\exp\Big(-\frac{1}{4\tau_2}((u_1\bar{\tau} - u_2)z - (u_1\tau - u_2)\bar{z})\Big) \\
\bar{\upsilon} &= \hat{\bar{\upsilon}}\exp\Big(+\frac{1}{4\tau_2}((u_1\bar{\tau} - u_2)z - (u_1\tau - u_2)\bar{z})\Big) \\
\rho &= \hat{\rho} + \frac{1}{4\tau_2}\Big((u_1\bar{\tau} - u_2)z + (u_1\tau - u_2)\bar{z}\Big)\,,
\end{aligned} \tag{3.3}$$

where the hatted fields are periodic.[4] The coset partition function then reads

$$Z_{cs}(\tau) = \int \mathcal{D}\hat{\phi}\mathcal{D}\hat{\upsilon}\mathcal{D}\hat{\bar{\upsilon}}\mathcal{D}\hat{\rho}\mathcal{D}b\mathcal{D}c \int_{-\infty}^{+\infty} du_1 du_2 e^{-S[\phi,\upsilon,\bar{\upsilon};\rho;b,c]}\,. \tag{3.4}$$

3.1 Ray-Singer torsion

The core of the computation uses the Ray-Singer analytic torsion [47], which arises from the path integral over $\hat{\upsilon}, \hat{\bar{\upsilon}}$. The relevant piece of the action, after substituting (3.3), is

$$S_{\upsilon,\bar{\upsilon}} = \left(\partial + \partial\hat{\phi} + \frac{1}{2\tau_2}(u_1\bar{\tau} - u_2)\right)\hat{\bar{\upsilon}}\left(\bar{\partial} + \bar{\partial}\hat{\phi} + \frac{1}{2\tau_2}(u_1\tau - u_2)\right)\hat{\upsilon}\,. \tag{3.5}$$

Note that the action is quadratic in $\hat{\upsilon}, \hat{\bar{\upsilon}}$. Following [27], we observe that we can disentangle the $\hat{\phi}$-dependence by a chiral rotation. The integral over $\upsilon, \bar{\upsilon}$ then becomes the regularised determinant of the Laplacian on a space of functions that have non-trivial holonomies around the cycles of the two-torus. Precisely this determinant was defined in [47] by using ζ-function regularisation. The regularised determinant is called the analytic torsion:[5]

$$\begin{aligned}
&\det\left|\partial + \frac{1}{2\tau_2}(u_1\bar{\tau} - u_2)\right|^{-2} = \\
&= \frac{(q\bar{q})^{-2/24}}{|\sin(\pi(u_1\tau - u_2))|^2}\frac{e^{\frac{2\pi}{\tau_2}(\mathrm{Im}(u_1\tau - u_2))^2}}{|\prod_{r=1}^{\infty}(1 - e^{2\pi i r\tau - 2\pi i(u_1\tau - u_2)})(1 - e^{2\pi i r\tau + 2\pi i(u_1\tau - u_2)})|^2}\,.
\end{aligned} \tag{3.6}$$

We introduced the usual notation $q = \exp(2\pi i\tau)$. The analytic torsion is periodic in the holonomies u_1 and u_2, as we would expect from gauge invariance.[6] If needed (for instance in order to check modular properties [47]), the analytic torsion can be re-written in terms of the θ_1-function.

[4] Trying to follow the holonomies of the gauge field through the field redefinitions we gave before gives rise to the difficulties we mentioned related to analytic continuation and reality of the fields. We chose the holonomies to be consistent with complex conjugation for υ, reality for ϕ, etc. We believe the resulting spectrum gives sufficient justification for this choice of analytic continuation.

[5] Note that the computation of the analytic torsion on the torus (cf. [47], p. 165-169), naturally resembles the usual computation of the partition function for a compact boson.

[6] It is also evident from the mathematical definition of analytic torsion in terms of a complex line bundle with non-trivial character $\chi(m\tau + n) = e^{2\pi i(mu_1 + nu_2)}$. Note that the authors of [2] appropriately use an analytically continued version of the analytic torsion that is not periodic.

3.2 Free contributions

In this subsection we treat the other contributions to the partition function which are basically the familiar free contributions, but some factors need to be treated with care. First of all note that there is a shift $k \to k-2$ in the kinetic term of $\hat{\phi}$ because of the contribution of the chiral rotation that we performed to disentangle ϕ and $\upsilon, \bar{\upsilon}$. The path integration over $\hat{\phi}$ and $\hat{\rho}$ will each give the usual periodic boson partition sum $\tau_2^{-1/2}|\eta(\tau)|^{-2}$ with overall factor $2\sqrt{k(k-2)}$. Moreover, the holonomy contributes an overall exponential factor. Finally, the contribution from the ghosts b, c that we introduced to gauge fix the U(1) symmetry is $\tau_2|\eta(\tau)|^4$ [48]. It is natural that the net effect of the gauge field is to cancel the free boson contribution to the $\mathrm{SL}(2,\mathbb{R})$ partition function.

3.3 Holonomies

It is convenient at this point to break the holonomy parameters u_1 and u_2 in integer and fractional parts, i.e. $u_1 = s_1 + w,\ u_2 = s_2 + m$ with $s_1, s_2 \in [0,1)$ and $w, m \in \mathbb{Z}$ running over the integers. Since the Ray-Singer torsion is periodic, it is only the overall exponential factor that depends on the integers w and m that parametrize the non-trivial windings for the compact bosons.

3.4 Combining ingredients

Combining all of the above we obtain for the modular invariant partition function:

$$
\begin{aligned}
Z_{cs}(\tau) &= 2(k(k-2))^{1/2} \int_0^1 ds_1 ds_2 \times \\
&\times \sum_{w,m=-\infty}^{+\infty} \frac{(q\bar{q})^{-2/24}}{|\sin(\pi(s_1\tau - s_2))|^2} \times \\
&\times \frac{e^{-\frac{k\pi}{\tau_2}|(s_1+w)\tau-(s_2+m)|^2 + \frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau-s_2))^2}}{|\prod_{r=1}^{\infty}(1-e^{2\pi i r\tau - 2\pi i(s_1\tau-s_2)})(1-e^{2\pi i r\tau+2\pi i(s_1\tau-s_2)})|^2}\,.
\end{aligned}
\tag{3.7}
$$

If we are interested in incorporating the coset theory as a factor in a string theory background $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1) \times \mathcal{M}$, we combine it with the modular invariant partition function $Z_{\mathcal{M}}$ for strings propagating on $\mathcal{M}$ and the reparametrization ghosts partition function Z_{ghosts}. We then integrate the modular parameter τ over the fundamental domain F_0 of the usual $\mathrm{SL}(2,Z)$ action on the complex τ-plane to obtain:

$$
\mathcal{Z} = \int_{F_0} \frac{d\tau d\bar{\tau}}{\tau_2} Z_{\mathcal{M}}(\tau) Z_{cs}(\tau) Z_{\text{ghosts}}(\tau)\,. \tag{3.8}
$$

The general form of the partition function corresponding to the background $\mathcal{M}$ is

$$
Z_{\mathcal{M}}(\tau) = (q\bar{q})^{-c_{\mathcal{M}}/24} \sum_i q^{h_i} \bar{q}^{\bar{h}_i}\,, \tag{3.9}
$$

where i labels all states of the CFT on $\mathcal{M}$ and $h_i, \bar{h}_i$ are the left-moving and right-moving conformal weights. Modular invariance implies that $h_i - \bar{h}_i$ is an integer. By $c_{\mathcal{M}}$ we denote the central charge of the CFT associated to $\mathcal{M}$. The total partition function can then be written as:

$$\begin{aligned}\mathcal{Z} &= 2(k(k-2))^{1/2}\int_{F_0}\frac{d\tau d\bar{\tau}}{\tau_2}\int_0^1 ds_1 ds_2 \times \\ &\times \sum_{w,m=-\infty}^{+\infty}\sum_i q^{h_i}\bar{q}^{\bar{h}_i} e^{4\pi\tau_2(1-\frac{1}{4(k-2)})-\frac{k\pi}{\tau_2}|(s_1+w)\tau-(s_2+m)|^2+2\pi\tau_2 s_1^2}\times \\ &\times\frac{1}{|\sin(\pi(s_1\tau-s_2))|^2}\left|\prod_{r=1}^{\infty}\frac{(1-e^{2\pi i r\tau})^2}{(1-e^{2\pi i r\tau-2\pi i(s_1\tau-s_2)})(1-e^{2\pi i r\tau+2\pi i(s_1\tau-s_2)})}\right|^2 .\end{aligned} \tag{3.10}$$

Now we need to disentangle the information hidden in this complicated formula.

4. Decomposition in characters

We want to connect our partition function computation to expectations from an algebraic analysis for the Hilbert space of the coset theory. To that end we need to manipulate our result further and determine the character contributions of the different affine representations to the partition function. In other words, we have to find the correct Hilbert space to trace over that will reproduce the above partition function. It is appropriate then to first recall some $SL(2,\mathbb{R})$ representation theory. See e.g. [18] for a more complete treatment. The representations of the affine algebras are the modules built on the $SL(2,\mathbb{R})$ representations using the creation modes of the currents. The $SL(2,\mathbb{R})$ representations we will encounter are the (principal) discrete representations with lowest weight $D_j^+ = \{|j,m\rangle : m = j, j+1, j+2, \ldots\}$ where the lowest weight state has J_0^3 eigenvalue $j>0$ and is annihilated by J_0^-, and similarly for the discrete highest weight representations $D_j^- = \{|j,m\rangle : m = j, j-1, j-2, \ldots\}$. The continuous representations $C_j^\alpha = \{|j,m\rangle : m = \alpha, \alpha\pm 1, \ldots\}$ where $\alpha\in[0,1)$, have an unbounded J_0^3 spectrum and $j=\frac{1}{2}+is$ with s real. The quadratic Casimir of all these representations is $c_2=-j(j-1)$.

After refreshing our memory on $SL(2,\mathbb{R})$ representations, we return to decompose the partition function into a sum over representations. We will do this in several steps. We first write the compact boson part in a more recognizable form. Secondly, we expand the partition function into a sum over states. And thirdly we identify contributions from discrete and continuous representations of $SL(2,\mathbb{R})$.

To work towards the spectrum predicted in [34], we first identify the momentum of the compact scalar. The relevant Poisson resummation is:

$$\sum_{m=-\infty}^{+\infty} e^{-\frac{k\pi}{\tau_2}(m^2-2m((s_1+w)\tau_1-s_2))} = \sqrt{\frac{\tau_2}{k}}\sum_{n=-\infty}^{+\infty} e^{-\frac{\pi\tau_2}{k}(n+\frac{ik}{\tau_2}((s_1+w)\tau_1-s_2))^2} \tag{4.1}$$

where we have resummed over m and the new integer $n\in\mathbb{Z}$ is the momentum of the scalar. Secondly, after the Poisson resummation, we expand the infinite products as well as the sin-prefactor in (3.10) into an infinite sum of exponential terms. For a state in the $SL(2,\mathbb{R})$ CFT with levels $N,\bar{N}$ and conformal weights $h,\bar{h}$ in the CFT on $\mathcal{M}$ (including reparametrization ghost contributions), the exponent arising from this expansion is:

$$\begin{aligned}\text{exponent}_{\text{expansion}} &= 2\pi i\tau_1(N+h-\bar{N}-\bar{h}+(q-\bar{q})s_1) - \\ &\quad -2\pi\tau_2(N+h+\bar{N}+\bar{h}+(q+\bar{q}+1)s_1) - 2\pi i s_2(q-\bar{q})\,, \end{aligned} \tag{4.2}$$

where q counts the number of $J^+_{n\leq 0}$ minus the number of $J^-_{n<0}$ operators, corresponding to the particular state under examination. A similar definition holds for $\bar{q}$ in terms of the right-moving creation operators. The overall contribution to the exponent is:

$$\begin{aligned}\text{exponent}_{\text{overall}} &= 4\pi\tau_2\Big(1-\frac{1}{4(k-2)}\Big) + 2\pi i n s_2 - \frac{\pi\tau_2}{k}n^2 - 2\pi i n\tau_1(w+s_1) + \\ &\quad +(2-k)\pi\tau_2 s_1^2 - 2k\pi\tau_2 s_1 w - k\pi\tau_2 w^2\,. \end{aligned} \tag{4.3}$$

Integrating over s_2 (see (4.2) and (4.3)) results in the constraint $q-\bar{q}=n$. After substituting $q-\bar{q}=n$, we find the total exponent

$$\begin{aligned}\text{exponent}_{\text{total}} &= 2\pi i\tau_1\Big(N+h-\bar{N}-\bar{h}-nw\Big) - \\ &\quad -2\pi\tau_2\Big(N+h+\bar{N}+\bar{h}+(q+\bar{q}+1)s_1 - \\ &\qquad -2\Big(1-\frac{1}{4(k-2)}\Big)+\frac{n^2}{2k}+\frac{k}{2}w^2+kws_1+\frac{k-2}{2}s_1^2\Big)\,. \end{aligned} \tag{4.4}$$

The integral over the first holonomy was fairly easy, and gave us one of the expected constraints [34]. It relates the momentum of the compact boson to the $J^3_0-\bar{J}^3_0$ eigenvalue in the $\mathrm{SL}(2,\mathbb{R})$ representation.

The integral over the second holonomy is far less trivial and needs some technical trickery, inspired by the analysis in [3]. It will allow us to separate the contributions from discrete and continuous representations of $\mathrm{SL}(2,\mathbb{R})$. We first introduce an auxiliary variable to incorporate a prefactor and the piece of the exponent quadratic in s_1:

$$\sqrt{(k-2)\tau_2}\,e^{-2\pi\tau_2(\frac{k-2}{2}s_1^2+(kw+1+(q+\bar{q}))s_1)} = \int_{-\infty}^{+\infty} dc\, e^{-\frac{\pi}{(k-2)\tau_2}c^2-2\pi(ic+\tau_2(kw+1+(q+\bar{q})))s_1}\,. \tag{4.5}$$

The integration over s_1 is now straightforward:

$$\begin{aligned}&\int_0^1 ds_1 e^{-2\pi s_1(ic+\tau_2(kw+1+(q+\bar{q})))} = \\ &\quad = \frac{-1}{2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}\Big(e^{-2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}-1\Big)\,. \end{aligned} \tag{4.6}$$

Combining it with the quadratic term in c results in the term

$$\frac{-1}{2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}\Big(e^{-\frac{\pi}{(k-2)\tau_2}c^2-2\pi(ic+\tau_2(kw+1+(q+\bar{q})))} - e^{-\frac{\pi}{(k-2)\tau_2}c^2}\Big)\,. \tag{4.7}$$

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4.1 Discrete representations

Now we observe that the exponent of the first term can be completed to a square if we set $c = 2\tau_2 s - i\tau_2(k-2)$. Shifting the contour of c (for the first term only) from $\mathrm{Im}\, c = 0$ to $\mathrm{Im}\, c = -i\tau_2(k-2)$, picks up residues from the poles of the denominator in the range $-\tau_2(k-2) < \mathrm{Im}\, c < 0$. The poles are located at $c = i\tau_2(kw + 1 + (q + \bar{q}))$ in the range:

$$-\tau_2(k-2) < \tau_2(kw + 1 + (q + \bar{q})) < 0 \,. \tag{4.8}$$

Now we note that we can interpret the pole contributions to the integral summed over $q, \bar{q}, w, n$, as the trace over a constrained Hilbert space. Consider the product Hilbert space $\hat{\mathcal{D}}_j^+$, the module built on the discrete representation $\mathcal{D}_j^+$ of $\mathrm{SL}(2,\mathbb{R})$, and the Hilbert space for the compact boson $\mathcal{H}^{\mathrm{U}(1)}$. The first constraint we put on the sum over states is $J_0^3 - \bar{J}_0^3 = n$, namely the constraint we obtained from the s_2-integration. The second constraint determines the quadratic Casimir j of the $\mathrm{SL}(2,\mathbb{R})$ representation in terms of the winding number of the compact boson: $J_0^3 + \bar{J}_0^3 = -kw$ or equivalently $kw + 1 + (q + \bar{q}) = 1 - 2j$. One way to see the necessity for this constraint is the fact that the T-duality $(J^3, \bar{J}^3, n, w) \to (\frac{1}{k}J^3, -\frac{1}{k}\bar{J}^3, -w, -n)$ is a symmetry of our partition function (which is reflected in the constraint equations).

The discrete nature of the representations is determined by the fact that the sin-prefactor gives rise to only one kind of operator at level zero, namely J_0^+, and not to J_0^- contributions. Notice moreover that we needn't sum over creation operators for the J^3-current or for the compact boson, since their contributions to the partition function were cancelled by the U(1) ghosts. The second constraint $kw + 1 + (q + \bar{q}) = 1 - 2j$, immediately implies (via 4.8) the expected bounds on j, the Casimir of the discrete representation:

$$\frac{1}{2} < j < \frac{k-1}{2} \,. \tag{4.9}$$

We emphasize that the upper bound we derived is not the one suggested in [34] but the improved bound[7] derived in [2] for the ungauged $\mathrm{SL}(2,\mathbb{R})$ WZW model.[8] Using the constraint we can rewrite the exponent in a familiar form. We obtain a sum over the described Hilbert space $\mathrm{Tr}_{\hat{\mathcal{D}}_j^+ \otimes \hat{\mathcal{D}}_j^+} q^{L_0^{cs}} \bar{q}^{\bar{L}_0^{cs}}$ where the L_0^{cs} operator takes the standard form:

$$L_0^{cs} = L_0^{\mathrm{SL}(2,\mathbb{R})} - L_0^{\mathrm{U}(1)} \,. \tag{4.10}$$

The conformal weights of the primary states, which agree with the total exponent after substitution of the values for the poles, are given by:

$$\begin{aligned} h_{cs} &= -\frac{j(j-1)}{k-2} + \frac{(n-kw)^2}{4k} \\ \bar{h}_{cs} &= -\frac{j(j-1)}{k-2} + \frac{(n+kw)^2}{4k} \,. \end{aligned} \tag{4.11}$$

[7]The improved bound was suggested for the ungauged model on the basis of consistency with the inclusion of spectral flowed representations in [1] and on the basis of fusion rules in [27]. The improved bound was shown to be necessary in the coset model for a tachyon free spectrum in Little String Theory in [40]. We prove this consistency requirement.

[8]As we will see in the following, a continuous spectrum opens up when j reaches either the lower or the upper bound [4, 5].

The summation is over states with the constraints $J_0^3 - \bar{J}_0^3 = n$, $J_0^3 + \bar{J}_0^3 = -kw$ and no contribution from the $J_{n<0}^3$ oscillators. Thus we interpreted the first part of our partition function as a character over a constrained product of an affine discrete $\mathrm{SL}(2,\mathbb{R})$ representation times a compact boson. We sum over discrete representations that satisfy the bound (4.9). For the parafermion interpretation of this Hilbert space we refer to [49, 1].

We remark that we crucially made use of the periodicity of the Ray-Singer torsion in the u_1-variable in our computation. If we would ignore this periodicity, it is clear from the analysis in [2] that we could identify the winding number w of the compact boson with the parameter w that controls the expansion of the different products in the denominator of the partition function, and therefore with the spectral flow parameter in the $\mathrm{SL}(2,\mathbb{R})$-WZW model.[9] This exemplifies in detail the relation uncovered in [1] between spectral flow and the winding of strings, in the coset model.

4.2 Continuous representations

We combine now the shifted integral over s of the first term in (4.7) with the integral over the second term, in which we rescale $c = 2\tau_2 s$. Including the summation over winding numbers w, we obtain

$$-\frac{1}{\pi}\sum_{w=-\infty}^{+\infty}\int_{-\infty}^{+\infty} ds \left[\frac{e^{-2\pi\tau_2(N+h+\bar{N}+\bar{h}-2+2\frac{s^2+1/4}{k-2}+\frac{n^2}{2k}+\frac{k}{2}(w+1)^2+(q+\bar{q}))}}{2is+k(w+1)-1+(q+\bar{q})} - \right.$$
$$\left. - \frac{e^{-2\pi\tau_2(N+h+\bar{N}+\bar{h}-2+2\frac{s^2+1/4}{k-2}+\frac{n^2}{2k}+\frac{k}{2}w^2)}}{2is+kw+1+(q+\bar{q})}\right]. \tag{4.12}$$

Inspecting the above expression, we observe that the first term of the $w-1$ sector and the second term of the w winding sector share the same exponent after spectral flow of the first by one unit $N, \bar{N} \to N+q, \bar{N}+\bar{q}$. The last operation is based on the isomorphism $\hat{\mathcal{D}}_j^{+,w-1} \cong \hat{\mathcal{D}}_{\frac{k}{2}-j}^{-,w}$ where the second upper index denotes the amount of spectral flow [1], i.e. these are discrete representations defined with respect to the algebra obtained after spectral flow. Combining terms this way we get

$$-\frac{1}{\pi}\sum_{w=-\infty}^{+\infty}\int_{-\infty}^{+\infty} ds\; e^{-2\pi\tau_2(N+h+\bar{N}+\bar{h}-2+2\frac{s^2+1/4}{k-2}+\frac{n^2}{2k}+\frac{k}{2}w^2)} \times$$
$$\times\left(\frac{1}{2is+kw-1+(q+\bar{q})} - \frac{1}{2is+kw+1+(q+\bar{q})}\right). \tag{4.13}$$

As in [2], these two terms can be interpreted as representing two halves of a continuous representation with $j = \frac{1}{2}+is$. The first term represents the contribution of a D^- representation (after spectral flow) and the second term still corresponds to a D^+ representation. In particular, note that the second term, when summed over states $(J_0^+\bar{J}_0^+)^r|\psi\rangle$ in $\mathcal{D}^+$, gives rise to a logarithmically divergent sum. We adopt here the regularisation procedure

[9] Note that this also follows from the fact that the integer holonomies w change the current algebra on the torus according to spectral flow. See e.g. [50].

of [2] and introduce a Liouville wall that cuts off the infinite volume otherwise available to the strings in the continuous representation.[10] Thus we obtain a regularised sum over the zeromodes of the following form:

$$-\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{A+r}e^{-r\epsilon}=-\frac{1}{2}\log\epsilon+\frac{1}{2}\frac{d}{dA}\log\Gamma(A),-\quad A=is+\frac{1}{2}(kw+1-n)\,. \tag{4.14}$$

Similarly, the first term in the integral can be interpreted as an infinite sum over states in a $\mathcal{D}^-$ Hilbert space of the form $(J_0^-\bar{J}_0^-)^r|\psi\rangle$

$$\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{B-r}e^{-r\epsilon}=-\frac{1}{2}\log\epsilon-\frac{1}{2}\frac{d}{dB}\log\Gamma(-B),\quad B=is+\frac{1}{2}(kw-1+n)\,. \tag{4.15}$$

The density of states as a function of s is then found to be

$$\rho(s)=\frac{1}{2\pi}2\log\epsilon+\frac{1}{2\pi i}\frac{d}{2ds}\log\frac{\Gamma(-is+\frac{1}{2}-m)\,\Gamma(-is+\frac{1}{2}+\bar{m})}{\Gamma(+is+\frac{1}{2}+\bar{m})\,\Gamma(+is+\frac{1}{2}-m)}\,, \tag{4.16}$$

where $m=\frac{1}{2}(n-kw)$, $\bar{m}=-\frac{1}{2}(kw+n)$ are the eigenvalues of J_0^3 and $\bar{J}_0^3$. In the above expression we have truncated the range of integration over s to $[0,\infty)$ using the invariance of the exponent under $s\to -s$. Thus, the contribution of the continuous representations of $\mathrm{SL}(2,\mathbb{R})$ combined with the momentum and winding modes of the free boson, can be written as

$$\sum_{w,n=-\infty}^{+\infty}\int_0^{+\infty}2ds\;\rho(s)\mathrm{Tr}_{\hat{\mathcal{C}}_{\frac{1}{2}+is}\otimes\hat{\mathcal{C}}_{\frac{1}{2}+is}}q^{L_0^{cs}}\bar{q}^{\bar{L}_0^{cs}} \tag{4.17}$$

where the conformal primaries have weights

$$\begin{aligned} h_{cs} &= \frac{s^2+\frac{1}{4}}{k-2}+\frac{(n-kw)^2}{4k}\\ \bar{h}_{cs} &= \frac{s^2+\frac{1}{4}}{k-2}+\frac{(n+kw)^2}{4k} \end{aligned} \tag{4.18}$$

and the trace over $\hat{\mathcal{C}}_{\frac{1}{2}+is}\otimes\hat{\mathcal{C}}_{\frac{1}{2}+is}$ is subject to the same constraints as before, namely $J_0^3+\bar{J}_0^3=-kw$, $J_0^3-\bar{J}_0^3=n$ and the J^3-current and free boson creation operators act trivially.

As in [2], we can perform a consistency check on the density of states by analysing the phase shift in a scattering experiment. We can introduce a Liouville wall for the continuous representation strings, to cut off the infinite volume available to them,[11] and relate the density of states to the phase shift for scattering a string in the bulk of $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$

[10] A rigorous justification of this procedure would require a precise identification of the coefficient of the exponential suppression after the introduction of a Liouville wall at a finite distance in the target space [2], and a precise treatment of the sum over the J_0^3 charge that is related to the creation and annihilation operators of the J^+ and J^- currents. As in [2], we will find justification for the adopted prescription from an independent scattering amplitude argument.

[11] In our case the volume divergence is apparent from the pole of the partition function (3.10) at $s_1 = 0 = s_2$. Excising the pole corresponds to introducing the Liouville wall.

and then off the Liouville wall [2]. Making use of the fact that the form of the scattering amplitude is the same for the coset theory as for the ungauged $\mathrm{SL}(2,\mathbb{R})$ model (see e.g. [40]), we can conclude that the density of states is indeed given by (4.16), where we obtain the eigenvalues of the J_0^3 and $\bar{J}_0^3$ operators from the constraint equations on the Hilbert space. This gives an overall consistency check on our regularisation procedure.

5. Conclusions

We have thereby finished the identification of the characters of the different representations in the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset partition function. We obtained agreement with the spectrum that one would get by imposing the usual [34] algebraic constraints on the spectrum derived in [1] for the ungauged model. We have thereby proved the correctness of this procedure from first principles. In particular we proved the upper bound on the spin and determined the density of string states in the continuous representations.

It is straightforward to extend our computation to the T-dual trumpet background [33, 34], since our analytic treatment is related smoothly to the algebraic treatment of the conformal field theory in which T-duality is manifest. The partition function of the Lorentzian black hole too, should now be within reach. It might also be possible to include a mass in the black hole background [35] and compute the temperature dependent partition function. This could provide a setup where aspects of stringy black hole thermodynamics could be addressed in a systematic manner.

The program of studying the coset theory can of course be followed along the lines of [3] by computing various two-, three- and four-point correlation functions to check the completeness of the Hilbert space. This should be straightforward due to the easy relations between the correlations functions for the parent and the coset theories. Note that we can interpret the winding number w and the momentum n as the winding of the string around the semi-infinite cigar, and the momentum around the circle at infinity. The winding number is not conserved since the string can slip off the semi-infinite cigar, but it is well possible that a precise restriction on the violation of winding number can be derived following [3] (appendix D).

One of our prime motivations for computing the partition function on the black hole background was to be able to analyse more rigorously the spectrum of string theory on backgrounds including the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset. Given our analysis this is now possible and should find applications, for example in analysing the holographic correspondence for these backgrounds.

It is also important to construct boundary states in the coset theory that would correspond to D-branes in these backgrounds. For the parent $\mathrm{SL}(2,\mathbb{R})$ conformal field theory, the construction was done in [51, 52, 53] and [55, 56, 57], with applications to AdS_3 string theory [58]–[61]. For the coset theory a similar construction should be possible [54, 55], and should yield information on D-branes in NS5-brane backgrounds (see e.g. [62, 63]).

Another extension of our results would be to analyse the partition function of the $N=2$ supersymmetric extension of our coset model and make connection with the work [64] (based on [65]). Namely, a precise analysis of the spectrum might shed additional light on the duality between the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset theory and the $N = 2$ Liouville CFT.

In summary, we believe that we provided a good basis for a precise path integral analysis of the spectrum of non-compact coset conformal field theories and for uncovering more secrets of black hole backgrounds in CFT and string theory.

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